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Weights for total division orderings on strings

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Abstract

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In this paper we consider total division orderings on strings. We give a simple proof of the fact that for each such ordering $\succ$ there exists an essentially unique, nontrivial set of weights such that if the weight of u is greater than the weight of v then $u \succ v$. It is known that all total division orderings on strings are rational, we prove a slightly stronger version of this result. Also, we use the ideas involved in the proof of the weights result to give a much simpler proof of the rationality result.

0. Introduction

The need to be able to prove termination of processes was recognized at least as far back as 1949 [14] when Turing discussed an algorithm for calculating integer factorials. It is known that termination is undecidable in general [21] but there is now great interest in techniques for proving termination of particular processes. A standard technique is to impose a well founded ordering on some aspect of the process. Most of the termination proofs in the literature use ad hoc methods based on variations of a few well known classes of orderings such as the Knuth–Bendix ordering [8], the recursive path ordering [4], and the polynomial orderings [1]. A survey of orderings which have been used for proving termination of rewriting

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systems is given in [5]. Recently we have begun to develop the theory of division orderings, which are known to be well founded by a result of Higman, [7]. The aim of such work is to provide a theoretical base upon which to build termination proofs.

In this paper we consider division orderings on strings, or equivalently on terms with one variable symbol and unary function symbols. The undecidability of termination for string rewriting systems is discussed in [20], definitions of the recursive path ordering for strings were given in [9] and [18], and many termination proofs for string rewriting systems have been carried out. For example, see [15] for a terminating rewriting system for the Jantzen monoid, [16] for an application to concurrency relations, [19] for applications in group theory of the Knuth–Bendix procedure for strings, and [2] for a general survey of string rewriting.

More recently, there have been significant developments which allow us to classify some of the termination orderings of interest, see [3, 12, 13]. This work has revealed the rich variety of termination orderings which are possible and the mathematical complexities which underly them. In the current paper we prove the following results.

Theorem A. *Given any total division ordering $\succ$ on a set $\{a_1,\ldots,a_n\}^*$ of strings there exists a nonzero weight function $f_\succ$ such that, for all strings u, v, if $f_\succ(u)>f_\succ(v)$ then $u\succ v$. Furthermore, the function $f_\succ$ is unique up to scalar multiplication. (A weight function is a map from $\{a_1,\ldots,a_n\}^*$ into the nonnegative real numbers with the property that for all strings, $a_{i_1}a_{i_2}\ldots a_{i_m}$, $f(a_{i_1}a_{i_2}\ldots a_{i_m})=f(a_{i_1})+f(a_{i_2})+\cdots+f(a_{i_m})$.)*

Theorem B. *Given any total division ordering $\succ$ on a set $\{a_1,\ldots,a_n\}^*$ of strings if u weakly dominates v through a letter with nonzero weight then $u\succ v$.*

In [3] it was proved that if the string u dominates the string v then $u\succ v$ for any total division ordering $\succ$. (See Section 4 below for definitions of domination and weak domination.) In this paper we prove that Theorem B above, which is a slightly stronger result than the one given in [3], is a consequence of Theorem A. In addition, the proof that we give of Theorem A motivates a direct proof of Theorem B; we also give this proof because it provides a simpler proof of the result first proved in [3].

After this paper was written it was pointed out to me by the referee that the existence of functions $f_\succ$ for total division orderings is a consequence of a theorem on compatible total orderings on free monoids proved by Saito et al. in [17]. A ordering on $\mathscr{A}^*$ is *compatible* if it satisfies (iii) of the definition of division orderings below. Theorem 3.4 of [17] states that every total compatible ordering on $\mathscr{A}^*$ has an associated nonzero weight function. However, there is no discussion of the uniqueness of these functions in [17] and no mention of the dominance results. We feel that it is worth publishing our proof of Theorem 2 below both because it provides a clear and easy to read proof of the existence of the functions $f_\succ$ in the case where total compatible orderings also satisfy (ii) below, and because it motivates the proof of the uniqueness of these functions, (Theorem 3 below).

1. Preliminaries

In this section we give the definitions and preliminary results that we shall need to prove the theorems.

Let $\mathscr{A}$ be a finite set whose elements we call *letters*, and let $\mathscr{A}^*$ denote the set of finite, possibly empty, sequences of letters from $\mathscr{A}$. We call the elements of $\mathscr{A}^*$ the *strings* on $\mathscr{A}$. A string v is a *substring* of a string u if v can be obtained by deleting zero or more letters from u. We denote the empty string in $\mathscr{A}^*$ by ε.

A *total division ordering* on $\mathscr{A}^*$ is a (strict) ordering $\succ$ that satisfies:

(i) For all $u, v \in \mathscr{A}^*$ either $u \succ v$, or $v \succ u$, or $u = v$.

(ii) $u \succ \varepsilon$, for all $u \in \mathscr{A}^*$ such that $u \neq \varepsilon$.

(iii) If $u \succ v$ then $xuy \succ xvy$, for all $x, y, u, v \in \mathscr{A}^*$.

A simple example of a division ordering on strings is the *length-then-lexicographic ordering*: a string is bigger than a shorter string and two strings of the same length are compared lexicographically.

We note the following property.

The substring property. *If $\succ$ is a division ordering then $u \succ v$ for all substrings v of u.*

Notation. Let $\#(w, a)$ denote the number of occurrences of the letter a in the string w.

We shall repeatedly use the following result proved by Martin [11].

Theorem (Martin). *Let $\succ$ be a total division ordering on $\mathscr{A}^*$. For each $u \in \mathscr{A}^*$ there exist permutations π, ρ such that*

$$a_{\pi(1)}^{t_{\pi(1)}} \cdots a_{\pi(n)}^{t_{\pi(n)}} \succeq u \succeq a_{\rho(1)}^{t_{\rho(1)}} \cdots a_{\rho(n)}^{t_{\rho(n)}},$$

where $t_i = \#(u, a_i)$.

To prove Theorem 2 (below) we need the following lemma about the approximation of real numbers by rationals. The proof that I originally gave used induction on the integer m. I am grateful to Mike Atkinson for pointing out the following proof using continued fractions.

Lemma 1. *Let τ be a positive irrational number.*

(a) *Given an integer $m \geqslant 0$ there exist positive integers p, q such that*

$$\frac{p}{q} > \tau > \frac{mp}{mq+1}.$$

(b) *Given an integer $m \geqslant 1$ there exist positive integers p, q such that*

$$\frac{mp}{mq-1} > \tau > \frac{p}{q}.$$

Proof. Let $q_0/p_0, q_1/p_1, \ldots$ be the rationals which are the convergents of the continued fraction approximation for $1/\tau$. (See, for example, Chapter X of [6] for a full discussion of continued fractions.) Then the properties of continued fractions imply that

$$\frac{q_{2n+1}}{p_{2n+1}} > \frac{1}{\tau} > \frac{q_{2n}}{p_{2n}}, \quad p_{n+1} > p_n, \quad \lim_{n\to\infty} \frac{q_n}{p_n} = \frac{1}{\tau}, \quad \text{for all } n \geqslant 1,$$

and, since $1/\tau$ is irrational, that the sequence of convergents is infinite. Thus we may choose an integer $k \geqslant 1$ such that $p_j > m$ for all $j \geqslant k$. Furthermore, from Theorem 171 of [6] we have

$$\frac{1}{p_{n+1}p_n} > \left| \frac{1}{\tau} - \frac{q_n}{p_n} \right|, \quad \text{for all } n.$$

(a) For $m=0$ we can find an integer g such that $g > \tau$. Then, since $\tau > 0$, we can choose $p = g$, $q = 1$. For $m \geqslant 1$ choose an integer k such that $p_j > m, j \geqslant k$. Then we have

$$\frac{1}{mp_{2k}} > \frac{1}{p_{2k+1}p_{2k}} > \left| \frac{1}{\tau} - \frac{q_{2k}}{p_{2k}} \right|.$$

Thus

$$\frac{1}{mp_{2k}} + \frac{q_{2k}}{p_{2k}} > \frac{1}{\tau} > \frac{q_{2k}}{p_{2k}},$$

and since $m > 0$

$$\frac{p_{2k}}{q_{2k}} > \tau > \frac{mp_{2k}}{mq_{2k}+1}.$$

so we take $p = p_{2k}$ and $q = q_{2k}$.

(b) The proof is almost identical to the proof of (a) except we take $p = p_{2k+1}$ and $q = q_{2k+1}$. $\square$

2. Invariants of a division ordering

In this section we shall prove the existence of certain invariants of division orderings on strings. The weights that we require for Theorem 2 are products of these invariants. We shall only prove the existence of the invariants for total division orderings. However a similar proof shows that they also exist for partial orderings, although in general they are not unique in the partial case.

Theorem 1. *Let $\succ$ be a total division ordering on $\mathscr{A}^*$. Given nonempty strings $u, v \in \mathscr{A}^*$ there exists a unique $\tau(u, v, \succ) \in \mathbf{R}^+ \cup \{0, \infty\}$ such that $u^i \succ v^j$ if $i > j\tau(u, v, \succ)$ and $v^j \succ u^i$ if*

$i<j\tau(u,v,\succ)$. *(If u is nonempty we could of course take $\tau(u,\varepsilon,\succ)$ to be 0 and $\tau(\varepsilon,u,\succ)$ to be ∞. However, $\tau(\varepsilon,\varepsilon,\succ)$ cannot exist.)*

Proof. We begin by proving $\tau(u,v,\succ)$ exists, then we prove it is unique.

Suppose first that for some r,s we have $u^r=v^s$. It is well known, see for example Proposition 1.3.1, p7 of [10], that there exists some $w\in\mathscr{A}^*$ such that $u=w^n$ and $v=w^m$. In this case we take $\tau(u,v,\succ)$ to be m/n. Then if $i/j>m/n>g/h$ we have

$$u^i=w^{in}\succ w^{jm}=v^j \quad\text{and}\quad v^h=w^{hm}\succ w^{gn}=u^g.$$

Now suppose that no power of u is equal to a power of v. Let

$$N=\{q\in\boldsymbol{Q}^+ \mid \text{for some } i,j,\ q=i/j \text{ and } u^i\succ v^j\}$$

$$M=\{q\in\boldsymbol{Q}^+ \mid \text{for some } i,j,\ q=i/j \text{ and } v^i\succ u^j\}.$$

If $N=\emptyset$ then we may take $\tau(u,v,\succ)=\infty$. If $M=\emptyset$ then we may take $\tau(u,v,\succ)=0$. We shall thus assume that both N and M are non-empty.

Suppose that $q\in N$ and $p\in M$. We show that $q>p$. By definition there exist positive integers i,j,n,m such that $u^i\succ v^j$ and $v^m\succ u^n$ where $q=i/j$ and $p=n/m$. Then we have that

$$v^{mi}\succ u^{ni}\succ v^{nj}, \quad\text{and so}\quad mi>nj.$$

Thus $q>p$ and the elements of N are upper bounds for the elements of M. Hence M has a least upper bound, $\tau\in\boldsymbol{R}$. Furthermore, since all the elements of N are upper bounds for M we have that τ is a lower bound for N.

We show that τ is the required number. If $r/s>\tau$ and $r/s\notin M$ since τ is an upper bound. Thus we cannot have $v^s\succ u^r$ and, since no power of u is a power of v and the ordering is total, we must have $u^r\succ v^s$. Similarly, if $r/s<\tau$ then $r/s\notin N$ so we must have $v^s\succ u^r$, as required.

Now suppose that we have $\rho(u,v,\succ)$ with the same property as $\tau(u,v,\succ)$. We shall assume that $\tau(u,v,\succ)>\rho(u,v,\succ)$. Then we can find r,s such that $\tau(u,v,\succ)>r/s>\rho(u,v,\succ)$. In which case we have both $u^r\succ v^s$ and $v^s\succ u^r$, which is a contradiction. Thus $\tau(u,v,\succ)$ is unique. $\square$

Note. For $u,v\in\mathscr{A}^*$, if $u\succ v$ then $1\in M$ and so $1\geqslant\tau(u,v,\succ)\geqslant 0$.

3. Weights of a total division ordering

In this section we shall prove the following theorem.

Theorem 2. *Let $\mathscr{A}=\{a_1,\dots,a_n\}$ and let $\succ$ be a total division ordering on $\mathscr{A}^*$. Then there exist nonnegative real numbers $\mu_1,\dots,\mu_n$, at least one of which is nonzero, such that*

$$\mu_n\#(u,a_n)+\cdots+\mu_1\#(u,a_1)>\mu_n\#(v,a_n)+\cdots+\mu_1\#(v,a_1)$$

implies that $u\succ v$.

Notice, without the requirement that at least one of the μ_i be nonzero this theorem would be trivially true.

As mentioned in the introduction to this paper, this theorem is a consequence of Theorem 3.4 of [17]. We give a different proof, in which we actually construct the μ_i from the invariants $\tau(a_{i+1},a_i,\succ)$. Then we show, see Theorem 3, that these weights are essentially unique.

Definition. A sequence $(v_n,\dots,v_1)$ of nonnegative real numbers is called a *sequence of weights* for the total division ordering $\succ$ on $\mathscr{A}^*$ if it has the property that

$$v_n\#(u,a_n)+\cdots+v_1\#(v,a_1)>v_n\#(v,a_n)+\cdots+v_1\#(v,a_1) \text{ implies } u\succ v,$$

where $a_n\succ\cdots\succ a_2\succ a_1$.

Then Theorem 2 says that there exists a nonzero sequence of weights for any total division ordering, and Theorem 3 (below) says that all the sequences of weights for $\succ$ are scalar multiples of $(\mu_n,\dots,\mu_1)$, i.e. that the μ_i are unique up to positive scalar multiplication.

Before giving the proofs we consider some examples. For the length-then-lexicographic ordering described in Section 1 we can take $\mu_i=1$, for $n\geqslant i\geqslant 1$.

Another well known set of examples are the recursive path orderings. In such an ordering the letters are ordered, say $a_n\succ\cdots\succ a_2\succ a_1$, then two strings are compared first on the number of occurrences of a_n, so if $\#(u,a_n)>\#(a_n,v)$ then $u\succ v$. Strings with the same numbers of a_n are compared on substrings which do not contain a_n. A precise definition of these orderings can be found in [9, 18, 3]. For these orderings $\mu_n=1$ and $\mu_i=0$, for $1\leqslant i\leqslant n-1$.

Given any sequence $\mu_1,\dots,\mu_n$ of nonnegative real numbers it is possible to construct a total division ordering $\succ$ which has the property given in Theorem 2. (So some permutation of $\mu_1,\dots,\mu_n$ is a sequence of weights for $\succ$.) This is done as follows. Define

$$\theta(u)=\mu_n\#(u,a_n)+\cdots+\mu_1\#(u,a_1),$$

then define $u\succ v$ if $\theta(u)>\theta(v)$. Now suppose that $\mu_{i_1}=\cdots=\mu_{i_m}=0$ and that $\mu_j\neq 0$ if $j\neq i_k$, for some k. If $\theta(u)=\theta(v)$, and if $\#(u,a_{i_k})=\#(v,a_{i_k})$ for $1\leqslant k\leqslant r-1$ but

$\#(u, a_{i_r}) > \#(v, a_{i_r})$ then set $u \succ v$. Finally, order the remaining unordered pairs lexicographically from the left. Rather tedious but essentially mechanical checking shows that this does indeed define a total division ordering. It clearly has the required property.

Proof of Theorem 2. Let $\mathscr{A} = \{a_1, \ldots, a_n\}$, let $\succ$ be a total division ordering on $\mathscr{A}^*$, and renaming the letters if necessary, assume that $a_n \succ \cdots \succ a_1$. Let $\tau_i = \tau(a_{i+1}, a_i, \succ)$ then, since $a_{i+1} \succ a_i$, we have $1 \geqslant \tau_i \geqslant 0$. Let $\mu_n = 1$ and $\mu_i = \tau_{n-1} \ldots \tau_i$, $n-1 \geqslant i \geqslant 1$, and define

$$\theta(u) = \mu_n \# (u, a_n) + \cdots + \mu_1 \# (u, a_1).$$

We show by induction on n that if $\theta(u) > \theta(v)$ then $u \succ v$.

If $n = 1$ we have that $\theta(u) = \#(u, a)$ and the result is a direct consequence of the substring property of division orderings.

Thus we assume that $n \geqslant 2$ and that the result is true for $n-1$. In particular, let $\succ'$ be the restriction of $\succ$ to $\mathscr{B}^*$, where $\mathscr{B} = \{a_1, \ldots, a_{n-1}\}$. Then $\tau'_i = \tau(a_{i+1}, a_i, \succ') = \tau_i$, $n-1 \geqslant i \geqslant 1$, so for $w \in \mathscr{B}^*$, $\mu'_{n-1} = 1$ and $\mu'_i = \tau_{n-2} \ldots \tau_i$, $n-1 > i \geqslant 1$, define

$$\theta'(w) = \mu'_{n-1} \# (w, a_{n-1}) + \cdots + \mu'_1 \# (w, a_1).$$

Then, since $w \succ' z$ implies that $w \succ z$, the induction hypothesis gives:

$$\text{for } w, z \in \mathscr{A}^* \text{ such that } \#(w, a_n) = 0 = \#(z, a_n), \text{ if } \theta'(w) > \theta'(z) \text{ then } w \succ z. \tag{$*$}$$

Suppose that we have $u, v \in \mathscr{A}^*$ such that $\theta(u) > \theta(v)$, and let $t_i = \#(u, a_i)$, $r_i = \#(v, a_i)$, $n \geqslant i \geqslant 1$. We consider three cases.

Case 1 (τ_{n-1} is rational): Assume that $\tau_{n-1} = p/q$, where $p \geqslant 0$, $q > 0$. Since $\theta(u) > \theta(v)$ we can find an integer $N > 0$ such that $q\theta(u) > 1/N + q\theta(v)$. Substituting p/q for τ_{n-1} and multiplying up by N we have

$$Nqt_n + Npt_{n-1} + \tau_{n-2} Npt_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npt_1$$

$$> 1 + Nqr_n + Npr_{n-1} + \tau_{n-2} Npr_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npr_1.$$

By Martin's theorem we can choose permutations π, ρ such that

$$u^{Np} \succeq a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_n^{Npt_n} \ldots a_{\pi(1)}^{Npt_{\pi(1)}}, \qquad a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_n^{Npr_n} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} \succeq v^{Np}.$$

The idea is to choose strings w, z which do not contain the letter a_n and such that $u^{Np} \succ w$ and $z \succ v^{Np}$. We then use the induction hypothesis to reason about w, z, i.e. we prove that $\theta'(w) > \theta'(z)$ and deduce that $w \succ z$.

Suppose first that $a_n^p \succ a_{n-1}^q$. By definition of τ_{n-1}, $a_{n-1}^{Nqr_n+1} \succ a_n^{Npr_n}$ so

$$u^{Np} \succeq a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_n^{Npt_n} \ldots a_{\pi(1)}^{Npt_{\pi(1)}} \succ a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_{n-1}^{Nqt_n} \ldots a_{\pi(1)}^{Npt_{\pi(1)}} = w, \quad \text{say},$$

$$v^{Np} \preceq a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_n^{Npr_n} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} \prec a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_{n-1}^{Nqr_n+1} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} = z, \quad \text{say}.$$

Then

$$\theta'(w) = (Nqt_n + Npt_{n-1}) + \tau_{n-2}Npt_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npt_1$$
$$> (1 + Nqr_n + Npr_{n-1}) + \tau_{n-2}Npr_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npr_1 = \theta'(z).$$

Thus, by (*), we have that $w \succ z$. Hence $u^{Np} \succ v^{Np}$, so $u \succ v$.

If $a_{n-1}^q \succ a_n^p$, then since $a_n^{Npt_n} \succ a_{n-1}^{Nqt_n-1}$ we have

$$u^{Np} \succeq a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_n^{Npt_n} \ldots a_{\pi(1)}^{Npt_{\pi(1)}} \succ a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_{n-1}^{Nqt_n-1} \ldots a_{\pi(1)}^{Npt_{\pi(1)}} = w, \quad \text{say},$$

$$v^{Np} \preceq a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_n^{Npr_n} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} \prec a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_{n-1}^{Nqr_n} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} = z, \quad \text{say}.$$

Then

$$\theta'(w) = (Nqt_n - 1 + Npt_{n-1}) + \tau_{n-2}Npt_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npt_1$$
$$> (Nqr_n + Npr_{n-1}) + \tau_{n-2}Npr_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npr_1 = \theta'(z).$$

Thus, by (*), we have that $w \succ z$. Hence $u^{Np} \succ v^{Np}$, so $u \succ v$ which proves the result in this case.

Case 2 (τ_{n-1} irrational and $r_n \geqslant t_n$): Since $\theta(u) > \theta(v)$ we have

$$t_{n-1} + \tau_{n-2}t_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 t_1$$
$$= \frac{\theta(u) - t_n}{\tau_{n-1}}$$
$$> \frac{\theta(v) - t_n}{\tau_{n-1}}$$
$$= \frac{r_n - t_n}{\tau_{n-1}} + r_{n-1} + \tau_{n-2}r_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 r_1.$$

Choose an integer $N > 0$ such that

$$t_{n-1} + \tau_{n-2}t_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 t_1$$
$$> \frac{1}{N} + \frac{r_n - t_n}{\tau_{n-1}} + r_{n-1} + \tau_{n-2}r_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 r_1,$$

and then, using Lemma 1(a), choose positive integers p, q such that

$$\frac{p}{q} > \tau_{n-1} > \frac{Nr_n p}{Nr_n q + 1}.$$

Then, by definition of τ_{n-1}, we have that

$$a_n^p \succ a_{n-1}^q \text{ and } a_{n-1}^{Nr_nq+1} \succ a_n^{Nr_np}.$$

By Martin's theorem we can choose permutations π, ρ such that

$$u^{Np} \succeq a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_n^{Npt_n} \ldots a_{\pi(1)}^{Npt_{\pi(1)}} \succ a_{\pi(n)}^{Npt_{\pi(n)}} \ldots a_{n-1}^{Nqt_n} \ldots a_{\pi(1)}^{Npt_{\pi(1)}} = w, \quad \text{say},$$

$$v^{Np} \preceq a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_n^{Npr_n} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} \prec a_{\rho(n)}^{Npr_{\rho(n)}} \ldots a_{n-1}^{Nqr_n+1} \ldots a_{\rho(1)}^{Npr_{\rho(1)}} = z, \quad \text{say}.$$

Again w, z do not contain the letter a_n so we can use the induction hypothesis to reason about them.

Since $r_n - t_n \geqslant 0$ and $1/\tau_{n-1} > q/p$ we have

$$\begin{aligned} &t_{n-1} + \tau_{n-2}t_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 t_1 \\ &\quad > \frac{1}{N} + \frac{q(r_n - t_n)}{p} + r_{n-1} + \tau_{n-2}r_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 r_1, \end{aligned}$$

and so

$$\begin{aligned} &(Nqt_n + Npt_{n-1}) + \tau_{n-2}Npt_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npt_1 \\ &\quad > (p + Nqr_n + Npr_{n-1}) + \tau_{n-2}Npr_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npr_1. \end{aligned}$$

Hence, since $p \geqslant 1$, we have

$$\begin{aligned} \theta'(w) &= (Nqt_n + Npt_{n-1}) + \tau_{n-2}Npt_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npt_1 \\ &> (1 + Nqr_n + Npr_{n-1}) + \tau_{n-2}Npr_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 Npr_1 = \theta'(z). \end{aligned}$$

Thus, by (∗), we have that $w \succ z$. Hence $u^{Np} \succ v^{Np}$, so $u \succ v$.

Case 3 (τ_{n-1} irrational and $t_n > r_n$): The argument is similar to that of Case 2 above. Since $\theta(u) > \theta(v)$, subtracting r_n from both sides and dividing by τ_{n-1}, we have

$$\begin{aligned} &\frac{t_n - r_n}{\tau_{n-1}} + t_{n-1} + \tau_{n-2}t_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 t_1 \\ &\quad > r_{n-1} + \tau_{n-2}r_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 r_1. \end{aligned}$$

Choose an integer $N > 0$ such that

$$\begin{aligned} &\frac{t_n - r_n}{\tau_{n-1}} + t_{n-1} + \tau_{n-2}t_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 t_1 \\ &\quad > \frac{1}{N} + r_{n-1} + \tau_{n-2}r_{n-2} + \cdots + \tau_{n-2} \ldots \tau_1 r_1. \end{aligned}$$

Since $t_n > 0$, using Lemma 1(b) we can choose positive integers p, q such that

$$\frac{Nt_n p}{Nt_n q - 1} > \tau_{n-1} > \frac{p}{q}.$$

Then, by definition of τ_{n-1}, we have that

$$a_{n-1}^q \succ a_n^p \text{ and } a_n^{Nt_n p} \succ a_{n-1}^{Nt_n q - 1}.$$

By Martin's theorem we can choose permutations π, ρ such that

$$u^{Np} \succeq a_{\pi(n)}^{Npt_{\pi(n)}} \dots a_n^{Npt_n} \dots a_{\pi(1)}^{Npt_{\pi(1)}} \succ a_{\pi(n)}^{Npt_{\pi(n)}} \dots a_{n-1}^{Nqt_n - 1} \dots a_{\pi(1)}^{Npt_{\pi(1)}} = w, \quad \text{say,}$$

$$v^{Np} \preceq a_{\rho(n)}^{Nr_{\rho(n)}} \dots a_n^{Npr_n} \dots a_{\rho(1)}^{Nr_{\rho(1)}} \prec a_{\rho(n)}^{Npr_{\rho(n)}} \dots a_{n-1}^{Nqr_n} \dots a_{\rho(1)}^{Npr_{\rho(1)}} = z, \quad \text{say.}$$

Since $t_n - r_n \geqslant 0$ and $q/p > 1/\tau_{n-1}$ we then have

$$\frac{q(t_n - r_n)}{p} - \frac{1}{N} + t_{n-1} + \tau_{n-2} t_{n-2} + \dots + \tau_{n-2} \dots \tau_1 t_1$$

$$> r_{n-1} + \tau_{n-2} r_{n-2} + \dots + \tau_{n-2} \dots \tau_1 r_1.$$

So

$$Nqt_n - Nqr_n - p + Npt_{n-1} + \tau_{n-2} Npt_{n-2} + \dots + \tau_{n-2} \dots \tau_1 Npt_1$$

$$> Npr_{n-1} + \tau_{n-2} Npr_{n-2} + \dots + \tau_{n-2} \dots \tau_1 Npr_1.$$

and hence, since $p > 0$,

$$\theta'(w) = (Nqt_n - 1 + Npt_{n-1}) + \tau_{n-2} Npt_{n-2} + \dots + \tau_{n-2} \dots \tau_1 Npt_1$$

$$> (Nqr_n + Npr_{n-1}) + \tau_{n-2} Npr_{n-2} + \dots + \tau_{n-2} \dots \tau_1 Npr_1 = \theta'(z).$$

Thus, by (*), we have that $w \succ z$. Hence $u^{Np} \succ v^{Np}$, so $u \succ v$, which proves the result. □

The examples above the proof of Theorem 2 show that any sequence of non-negative reals is a sequence of weights for some total division ordering, and it is clear that the zero sequence is a sequence of weights for any division ordering. However, even a nonzero sequence does not completely determine the ordering. For example there are uncountably many distinct total division orderings on $\{a_1, a_2\}^*$ generated by matrices[1], and all of these have $(1, 0)$ as a sequence of weights. Another example is the length-then-right-lexicographic ordering, in which strings of the same length are compared lexicographically from the right rather than the left. It has the same weights as the length-then-lexicographic ordering, $\mu_i = 1$, $n \geqslant i \geqslant 1$. However, the next theorem

[1] See [13] for the definition of matrix generated orderings.

shows that the converse *is* true: the sequence of weights of a division ordering is unique up to scalar multiplication.

Theorem 3. *Let $\succ$ be a total division ordering on $\mathscr{A}^*$. Then $(v_n, \ldots, v_1)$ is a sequence of weights for $\succ$ if and only if there exists a nonnegative real number r such that $v_i = r\mu_i$, $1 \leqslant i \leqslant n$. (Here, as in the proof of Theorem 2 we define $\mu_n = 1$ and $\mu_i = \mu_{i+1}\tau(a_{i+1}, a_i, \succ)$, where $a_n \succ \cdots \succ a_1$.)*

Proof. Define

$$\theta(u) = \mu_n \#(u, a_n) + \cdots + \mu_1 \#(u, a_1), \qquad \psi(u) = v_n \#(u, a_n) + \cdots + v_1 \#(u, a_1).$$

First we show that for any nonnegative real r, $(v_n, \ldots, v_1) = (r\mu_n, \ldots, r\mu_1)$ is a sequence of weights for $\succ$. If $r = 0$ then the result is trivial because the condition required is vacuous. If $r > 0$ we have that $r\theta(u) = \psi(u) > \psi(v) = r\theta(v)$ implies that $\theta(u) > \theta(v)$. Then the proof of Theorem 2 shows that if $\theta(u) > \theta(v)$ then $u \succ v$, as required.

Now suppose that $(v_n, \ldots, v_1)$ is a sequence of weights for $\succ$. We show by contradiction that $v_i = v_n\mu_i$, $1 \leqslant i \leqslant n$, and thus that $r = v_n$.

Suppose that for some k, $v_k \neq v_n\mu_k$. Note that, since $\mu_n = 1$, we have $k \leqslant n-1$. Furthermore, if $v_n = 0$ then $\psi(a_k) = v_k > 0 = \psi(a_n)$ implying $a_k \succ a_n$, which is a contradiction. Thus we have that $v_n > 0$. The proof now splits into two cases.

If $v_k/v_n > \mu_k$ choose positive integers r, s such that

$$\frac{v_k}{v_n} > \frac{r}{s} > \mu_k.$$

Then we have that

$$\psi(a_k^s) = sv_k > rv_n = \psi(a_n^r) \quad \text{and} \quad \theta(a_n^r) = r > s\mu_k = \theta(a_k^s).$$

Since $(v_n, \ldots, v_1)$ and $(1, \mu_{n-1}, \ldots, \mu_1)$ are both sequences of weights for $\succ$ this would give $a_k^s \succ a_n^r$ and $a_n^r \succ a_k^s$, which is a contradiction.

The case $v_k/v_n < \mu_k$ results in a contradiction in a similar way. Thus we must have $v_k = v_n\mu_k$ for all k, as required. □

4. Domination results

In [3] it was shown that all total division orderings on strings are rational (see below for the definition of a rational ordering). In this section we prove a slightly stronger result, and show that it has the rationality result as a trivial consequence. This stronger result was first discovered as a corollary of Theorem 2 above, and we give a proof of it in this way. However, there is also a direct proof which is only slightly more complicated, and we give this proof as well as it provides a simpler proof of the result in [3].

Definition. We say that u *weakly dominates* v *through* b if for all $a \in \mathscr{A}$ we have $\#(u,a) \geqslant \#(v,a)$ and if $\#(u,b) > \#(v,b)$.

Notation. For $u \in \mathscr{A}^*$ we let $\mathscr{A}(u) = \{a \in \mathscr{A} \mid \#(u,a) > 0\}$.

Theorem 4. *Let $\mathscr{A} = \{a_1, \ldots, a_n\}$ and let $\succ$ be a total division ordering on $\mathscr{A}^*$. Suppose that u weakly dominates v through b and that for each $a \in \mathscr{A}(v)$ there exists j such that $b^j \succ a$. Then we must have that $u \succ v$.*

Proof. Suppose that u weakly dominates v through b and that for all $a \in \mathscr{A}(v)$ there exists an integer j such that $b^j \succ a$. We may, and we shall, assume that $j \geqslant 2$. By considering a subword of u if necessary, we may assume that $\mathscr{A}(u) = \mathscr{A}(v) \cup \{b\}$. Choose r maximal such that $a_r \in \mathscr{A}(u)$ and let $\mathscr{B} = \{a_r, \ldots, a_1\}$, where $a_r \succ \cdots \succ a_1$. So $u, v \in \mathscr{B}^*$. Let $\succ'$ denote the restriction of $\succ$ to $\mathscr{B}^*$, let $\tau_i = \tau(a_{i+1}, a_i, \succ') = \tau(a_{i+1}, a_i, \succ)$, let $\mu'_r = 1$, let $\mu'_i = \tau_{r-1} \ldots \tau_i$ and, for $w \in \mathscr{B}^*$, let

$$\theta'(w) = \#(w, a_r) + \mu'_{r-1} \#(w, a_{r-1}) + \cdots + \mu'_1 \#(w, a_1).$$

So by the proof of Theorem 2, $(\mu'_r, \ldots, \mu'_1)$ is a sequence of weights for $\succ'$, and $\theta'(u) > \theta'(v)$ implies that $u \succ' v$ and hence that $u \succ v$. Thus we show that $\theta'(u) > \theta'(v)$.

Since $a_r \in \mathscr{A}(u) = \mathscr{A}(v) \cup \{b\}$, we either have $a_r \in \mathscr{A}(v)$ or $a_r = b$. In either case, since $j \geqslant 2$, we have by assumption that $b^j \succ a_r$. We also have that $b = a_i$, for some i such that $1 \leqslant i \leqslant r$. Thus, for k such that $r-1 \geqslant k \geqslant i$ we have

$$a_k^j \succeq b^j \succ a_r \succeq a_{k+1}.$$

Thus $\tau_k > 0$, for $r-1 \geqslant k \geqslant i$, and hence $\mu'_i > 0$. Since $\#(u, a_l) \geqslant \#(v, a_l)$ for all l, and $\#(u, a_i) > \#(v, a_i)$ we have $\theta'(u) > \theta'(v)$, as required. $\square$

Note. For any division ordering on $\mathscr{A}^*$ and any $u \in \mathscr{A}^*$, if u weakly dominates v through a letter a with $a \succeq$b for all $b \in \mathscr{A}(v)$ then $u \succ v$.

We now define what it means for an ordering to be rational and show that it is a trivial consequence of the above Note that all total division orderings are rational.

Definition. A string u *dominates* a string v if $u \neq \varepsilon$ and if $\#(u,a) > \#(v,a)$, for all $a \in \mathscr{A}(v)$. An ordering $>$ on $\mathscr{A}^*$ *is rational* if u dominates v implies that $u > v$.

Suppose that $\succ$ is a total division ordering and that u dominates v. Let $\mathscr{A}(u) = \{b_r, \ldots, b_1\}$ where $b_r \succ \cdots \succ b_1$. Then we have that u weakly dominates v through b_r. Since u dominates v we have $\mathscr{A}(v) \subseteq \mathscr{A}(u)$, thus $b_r \succeq a$, for all $a \in \mathscr{A}(v)$. Thus the result follows from the Note above.

As it stands, the above proof is no simpler than the proof given in [3] because it relies, via Theorem 4, on the proof of Theorem 2. Thus we now give a direct proof of Theorem 4.

A direct proof of Theorem 4. We may suppose without loss of generality that $\#(u,b)=\#(v,b)+1$ and that $\#(u,a_i)=\#(v,a_i)$ if $a_i \neq b$. We let $\mathscr{A}(v)\cup\{b\}=\{b=b_1,b_2,\ldots,b_k\}$.

We assume for contradiction that $v \succeq u$. Then we also have that $v^k \succeq u^k$. Let $m=\#(u^k,b)=\#(v^k,b)+k$ and $\#(u^k,b_i)=\#(v^k,b_i)=m_i$ for $2 \leqslant i \leqslant k$. By Martin's result there exist permutations π and ρ of $\{1,\ldots,k\}$ such that

$$b_{2\pi}^{m_{2\pi}}\ldots b^{m-k}\ldots b_{k\pi}^{m_{k\pi}} \succeq v \succeq u \succeq b_{2\rho}^{m_{2\rho}}\ldots b^{m}\ldots b_{k\rho}^{m_{k\rho}}.$$

Since $b_i \in \mathscr{A}(v)$ so we can choose j_i such that

$$b^{j_i} \succ b_i^{m_i} \succ b^{j_i-1}.$$

Then we have that

$$b^{j_{2\pi}}\ldots b^{m-k}\ldots b^{j_{k\pi}} \succ b_{2\pi}^{m_{2\pi}}\ldots b^{m-k}\ldots b_{k\pi}^{m_{k\pi}} \succeq b_{2\rho}^{m_{2\rho}}\ldots b^{m}\ldots b_{k\rho}^{m_{k\rho}}$$
$$\succ b^{j_{2\rho}-1}\ldots b^{m}\ldots b^{j_{k\rho}-1}.$$

Of course, we have

$$\sum_{i=2}^{k} j_{i\pi}=\sum_{i=2}^{k} j_{i\rho}=N, \text{ say,}$$

and thus

$$b^{N+m-k} \succ b^{N-k+1+m}$$

which is the required contradiction. $\square$

The above results suggest that it is worth studying the numbers $\tau(u,v,\succ)$ in more detail. However, here we just content ourselves with proving one interesting result.

Theorem 5. *Let $\succ$ be a total division ordering on $\mathscr{A}^*$. For all nonempty strings u, w and all strings v such that $\tau(u,v,\succ), \tau(v,w,\succ) \neq \infty$, we have $\tau(u,w,\succ)=\tau(u,v,\succ)\tau(v,w,\succ)$.*

Proof. The condition $\tau(u,v,\succ), \tau(v,w,\succ) \neq \infty$ just implies that for some $i,j,k,l>0$ we have $u^i \succ v^j$ and $v^k \succ w^l$.

If $\tau(v,w,\succ)=0$ then $v \succ w^m$, for all m. If $w^m \succ u$ then we have $v^j \succ w^{mi} \succ u^i$, which is a contradiction. Thus $u \succ w^m$ for all m and $\tau(u,w,\succ)=0$. Similarly, if $\tau(u,v,\succ)=0$ then $u \succ v^m$, for all m; so $u^l \succ v^{km}$, for all m. If $w^m \succ u$ then we have $w^{lm} \succ u^l \succ v^{km}$, which is a contradiction. Thus again $\tau(u,w,\succ)=0$. Hence we may assume that $\tau(v,w,\succ)$, $\tau(u,v,\succ)>0$.

Suppose that $\tau(u,w,\succ)>\tau(u,v,\succ)\tau(v,w,\succ)$. Then we can choose positive integers p,q such that

$$\tau(u,w,\succ)>\frac{p}{q}>\tau(u,v,\succ)\tau(v,w,\succ).$$

Thus $w^q \succ u^p$. Furthermore, we can then choose positive integers g,h such that

$$\frac{p}{q\tau(v,w,\succ)}>\frac{g}{h}>\tau(u,v,\succ).$$

Then

$$\frac{ph}{qg}>\tau(v,w,\succ),$$

and $u^g \succ v^h$ and $v^{ph} \succ w^{qg}$. Hence

$$v^{ph} \succ w^{qg} \succ u^{pg} \succ v^{ph},$$

which is a contradiction.

In the case $\tau(u,w,\succ)<\tau(u,v,\succ)\tau(v,w,\succ)$ we get a contradiction in a similar fashion. Thus we must have $\tau(u,w,\succ)=\tau(u,v,\succ)\tau(v,w,\succ)$ as claimed. $\square$

Corollary. For μ_i as in the proof of Theorem 2, we have $\mu_i=\tau(a_n,a_i,\succ)$, $1\leqslant i\leqslant n$.

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